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CONSTRUCTION OF THE ADJOINT PROBLEM TO THE DISCRETE PROBLEMS FOR THE SECOND ORDER EQUATION

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Abstract. The paper is devoted to the construction of the adjoint boundary value problem to the boundary value problems posed for the second order differential equation with discrete additive derivatives. The differences are given between presented scheme and the usual scheme, with the help of which the adjoint operator to the boundary value problems for the ordinary linear differential equation is constructed.

Keywords: discrete additive derivative, discrete boundary value problem, adjoint problem.
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1 Introduction

As is known discrete processes are less studied comparatively to the continuous ones. Despite the fact that many physical phenomena, for example, wave propagation has both continuous and discrete character (Dirac, 1932; Misha, 1978) and in classical mathematics the arithmetic progression and Fibonacci number (Vorobyev, 1984; Gelfond, 1967) are mainly known. The adjoint method is an elegant approach for the computation of the gradient of a cost function to identify a set of parameters. An additional set of differential equations has to be solved to compute the adjoint variables, which are further used for the gradient computation. However, the accuracy of the numerical solution of the adjoint differential equation has a great impact on the gradient. Hence, an alternative approach is the discrete adjoint method, where the adjoint differential equations are replaced by algebraic equations (Laub et al., 2018). Discrete adjouint problems are also met in investigation of the turbulent flow problems (Yang et al., 2019; He et al., 2018). The known results in the study of the ordinary linear differential equation for the difference equations are given for example in (Izadi et al., 2009). The results of some our investigations in this direction were given in (Izadi et al., 2009; Aliyev et al., 2020).

Here we construct adjoint boundary value problems to the boundary value problem posed for the ordinary linear differential equation of the second order with discrete additive derivatives. It would useful if we first consider the work (Hassani & Aliyev, 2008).

2 Statement of the problem

Let's consider the following boundary-value problem:

$$ly_n \equiv \; y_n^{('')} + ay_n^{(')} + by_n = f_n, \;\; 0 \leq n \leq N-2, \tag{1}$$

$$\begin{aligned} y_N + dy_0 = 0, \\ y_{N-1}{+}\beta y_1 = 0. \end{aligned} \tag{2}$$

where a, b, α and β are given real numbers; f_n is the given sequence; y_n is the sought sequence.

We define the discrete additive derivative by the relation

$$y_n^{(')} = y_{n+1} - y_n, \tag{3}$$

and the discrete additive integral by the relation

$$\int_0^n f_k = \sum_{k\ =0}^{n-1} f_k. \tag{4}$$

Multiplying the left side ly_n of equation (1) by

$$\tilde{l}Z_n \equiv C_2 Z_n^{('')} + C_1 Z_n^{(')} + C_0 Z_n, \tag{5}$$

where C_2, C_1 and C_0 are arbitrary constants, we get

$$ly_n\tilde{l}Z_n = C_2 y_n^{('')} Z_n^{('')} + C_1 y_n^{('')} Z_n^{(')} + C_0 y_n^{('')} Z_n + aC_2 y_n^{(')} Z_n^{('')} +$$

$$+aC_1 y_n^{(')} Z_n^{(')} + aC_0 y_n^{(')} Z_n + bC_2 y_n Z_n^{('')} + bC_1 y_n Z_n^{(')} + bC_0 y_n Z_n. \tag{6}$$

It is known that

$$(y_n Z_n)^{(')} = y_n^{(')} Z_n^{(')} + y_n^{(')} Z_n + y_n Z_n^{(')},$$

$$(y_n^{(')} Z_n)^{(')} = \ y_n^{('')} Z_n^{(')} + y_n^{('')} Z_n + y_n^{(')} Z_n^{(')},$$

$$(y_n Z_n^{(')})^{(1)} \ = y_n^{(')} Z_n^{('')} + y_n^{(')} Z_n^{(')} + y_n Z_n^{('')},$$

and

$$(y_n^{(')} Z_n^{(')})^{(')} = y_n^{('')} Z_n^{('')} + y_n^{('')} Z_n^{(')} + y_n^{(')} Z_n^{('')}.$$

Then

$$y_n^{(')} Z_n^{(')} = (y_n Z_n)^{(')} - y_n^{(')} Z_n - y_n Z_n^{(')},$$

$$y_n^{('')} Z_n^{(')} = (y_n^{(')} Z_n)^{(')} - -y_n^{('')} Z_n - y_n^{(')} Z_n^{(')} = (y_n^{(')} Z_n)^{(')} -$$

$$-y_n^{('')} Z_n - (y_n Z_n)^{(')} + y_n^{(')} Z_n + y_n Z_n^{(')},$$

$$y_n^{(')} Z_n^{('')} = (y_n Z_n^{(')})^{(')} - y_n^{(')} Z_n^{(')} - y_n Z_n^{('')} =$$

$$= (y_n Z_n^{(')})^{(')} - (y_n Z_n)^{(')} + y_n Z_n^{(')} - y_n Z_n^{('')},$$

$$y_n^{('')} Z_n^{('')} = (y_n^{(')} Z_n^{(')})^{(')} - y_n^{('')} Z_n^{(')} - y_n^{(')} Z_n^{('')} =$$

$$= (y_n^{(')} Z_n^{(')})^{(')} - \left(y_n^{(')} Z_n\right)^{(')} + (y_n Z_n)^{(')} + y_n^{('')} Z_n - y_n^{(')} Z_n - y_n Z_n^{(')} -$$

$$-(y_n Z_n^{(')})^{(')} + (y_n Z_n)^{(')} - y_n^{(')} Z_n - y_n Z_n^{(')} + y_n Z_n^{('')} =$$

$$= (y_n^{(')} Z_n^{(')})^{(')} - \left(y_n^{(')} Z_n\right)^{(')} - (y_n Z_n^{(')})^{(')} + 2(y_n Z_n)^{(')} + y_n^{('')} Z_n -$$

$$2y_n^{(')} Z_n - 2y_n Z_n^{(')} + y_n Z_n^{('')}.$$

Considering the obtained relations in (6), we get

$$ly_n\tilde{l}Z_n = C_2(y_n^{(')}Z_n^{(')})^{(')} - C_2(y_n^{(')}Z_n)^{(')} - C_2(y_nZ_n^{(')})^{(')} + 2C_2(y_nZ_n)^{(')}+$$

$$+C_2y_n^{('')}Z_n - 2C_2y_n^{(')}Z_n - 2C_2y_nZ_n^{(')} + C_2y_nZ_n^{('')} + C_1(y_n^{(')}Z_n)^{(')}-$$

$$-C_1(y_nZ_n)^{(')} - C_1y_n^{('')}Z_n + C_1y_n^{(')}Z_n + C_1y_nZ_n^{(')} + C_0y_n^{('')}Z_n+$$

$$+aC_2(y_nZ_n^{(')})^{(')} - aC_2(y_nZ_n)^{(')} + aC_2y_n^{(')}Z_n + aC_2y_nZ_n^{(')} - aC_2y_nZ_n^{('')}+$$

$$+aC_1(y_nZ_n)^{(')} - aC_1y_n^{(')}Z_n - aC_1y_nZ_n^{(')} + aC_0y_n^{(')}Z_n + bC_2y_nZ_n^{('')}+$$

$$+bC_1y_nZ_n^{(')} + bC_0y_nZ_n = C_2(y_n^{(')}Z_n^{(')})^{(')} + (C_1 - C_2)(y_n^{(')}Z_n)^{(')}+$$

$$+(aC_2 - C_2)(y_nZ_n^{('')})^{(')} + (2C_2 - C_1 - aC_2 + aC_1)(y_nZ_n)^{(')}+ \tag{7}$$

$$+\left(C_2 - C_1 + C_0\right)y_n^{('')}Z_n + \left(C_1 - 2C_2 + aC_2 - aC_1 + aC_0\right)y_n^{(')}Z_n+$$

$$+\left(C_2 - aC_2 + bC_2\right)y_nZ_n^{('')} + \left(C_1 - 2C_2 + aC_2 - aC_1 + bC_1\right)y_nZ_n^{(')} + bC_0y_nZ_n =$$

$$= C_2(y_nZ_n)^{('')} + (C_1 - 2C_2)(y_n^{(')}Z_n)^{(')} + (aC_2 - 2C_2)(y_nZ_n^{(')})^{(')}+$$

$$+(2C_2 - C_1 - aC_2 + aC_1)(y_nZ_n)^{(')} + \left(C_2 - C_1 + C_0\right)y_n^{('')}Z_n+$$

$$+(C_1 - 2C_2 + aC_2 - aC_1 + aC_0)y_n^{(')}Z_n + (C_2 - aC_2 + bC_2)y_nZ_n^{('')}+$$

$$+(C_1 - 2C_2 + aC_2 - aC_1 + bC_1)y_nZ_n^{(')} + bC_0y_nZ_n.$$

We choose arbitrary constants C_2, C_1 and C_0 to provide the absence of the terms $y_n^{('')}Z_n$ and $y_n^{(')}Z_n$ in the right hand side of (7), i.e.

$$\begin{array}{l} C_2 - C_1 + C_0 = 0, \\ C_1 - 2C_2 + aC_2 - aC_1 + aC_0 = 0. \end{array} \tag{8}$$

Then from (8) we get

$$C_1 = C_0 + C_2$$

and

$$C_0 + C_2 - 2C_2 + aC_2 - aC_0 - aC_2 + aC_0 = 0.$$

Thus, we have

$$C_2 = C_0 = 1, \qquad C_1 = 2. \tag{9}$$

Then (7) turns to

$$ly_n\tilde{l}Z_n = (y_nZ_n)^{('')} + (a-2)(y_nZ_n^{(')})^{(')}+$$

$$+a(y_nZ_n)^{(')} + \left(1 - a + b\right)y_nZ_n^{('')}+ \tag{10}$$

$$+(2b - a)y_nZ_n^{(')} + by_nZ_n$$

Summing up the resulting expression from zero to $N-2$, we get

$$\int_0^{N-1} ly_n\tilde{l}Z_n = (y_{N-1}Z_{N-1})^{(')} - (y_0Z_0)^{(')}+$$

$$+(a-2)\left[y_{N-1}Z_{N-1}^{(')}-y_0\ Z_0^{(')}\right]+\ \text{a}\ \left[y_{N-1}Z_{N-1}-y_0\ Z_0\right]+$$

$$\int_0^{N-1} y_n\left[(1-a+b)\,Z_n^{('')}+(2b-a)\,Z_n^{(')}+b\ Z_n\right] \tag{11}$$

Thus, instead of the Lagrange's formula, known from the theory of linear operators (Hassani & Aliyev, 2008), we get formula (11), i.e. the conjugate to (1) equation is given by

$$l^*Z_n\equiv(1-a+b)\,Z_n^{''}+(2b-a)\,Z_n^{(')}+bZ_n=f_n\ ,\qquad 0\le n\le N-2. \tag{12}$$

Now we formulate the boundary condition for the adjoint problem.

For this purpose first we separate the non-integrant terms (i.e. linear expressions) from the Lagrange's analogue formula

$$\begin{gathered}y_NZ_N-y_{N-1}Z_{N-1}-y_1Z_1+y_0Z_0+(a-2)\left[y_{N-1}\left(Z_N-Z_{N-1}\right)-\right.\\ \left.-y_0(Z_1-Z_0\ )\right]+a\left(y_{N-1}Z_{N-1}-y_0Z_0\right)=\\ =y_NZ_N-y_{N-1}\left[Z_{N-1}-(a-2)\left(Z_N-Z_{N-1}\right)-a\ Z_{N-1}\right]-\\ -y_1Z_1+y_0\left[Z_0-(a-2)\left(Z_1-Z_0\right)-a\ Z_0\right]=\\ =y_NZ_N-Y_{N-1}\left[-(a-2)\ Z_N\ -Z_{N-1}\right]-y_1Z_1+y_0\left[(2-a)Z_1-\ Z_0\right].\end{gathered} \tag{13}$$

Now, in boundary conditions (2) we define y_N , y_{N-1} and substitute them into (13)

$$\begin{gathered}y_NZ_N-y_{N-1}\left[-\left(a-2\right)Z_N\ -Z_{N-1}\right]-y_1Z_1+y_0[(2-a)\,Z_1-Z_0=\\ =-\alpha y_0Z_N+\ \beta y_1\left[-\left(a-2\right)Z_N\ -Z_{N-1}\right]-\\ -y_1Z_1+y_0\left[(2-a)\,Z_1-\ Z_0\right]=-y_1\left[Z_1+\beta\left(a-2\right)Z_N+\beta\ Z_{N-1}\right]+\\ +y_0\left[(2-a)\,Z_1-Z_0-\alpha\ Z_N\right].\end{gathered} \tag{14}$$

Finally, we obtain the following boundary condition to the adjoint problem.

$$\begin{cases}\beta\,(a-2)\,Z_N+\beta Z_{N-1}+Z_1=0,\\ \quad\alpha Z_N+(a-2)\,Z_1+Z_0=0.\end{cases} \tag{15}$$

Thus, we get the following statement.

Theorem 1. *Let us assume that a, b,α and β are given real constants, f_n is a given sequence. Then adjoint to* (1)-(2) *problem is given by formula* (12) *and* (15).

Now let us find the fundamental solution of equation (12). The partial solution of the corresponding homogeneous equation

$$1-a+b)Z_n^{('')}+(2b\ -a)Z_n^{(')}+bZ_n=0, \tag{16}$$

we seek in the form

$$Z_n=(\vartheta+1)^n. \tag{17}$$

Substituting (17) into (16), we get the following characteristic equation

$$(1-a+b)\vartheta^2+(2b-a)\,\vartheta+b=0, \tag{18}$$

$$\vartheta = \frac{(a-2b)+(-1)^k\sqrt{(a-2b)^2-4b(1-a+b)}}{2(1-a+b)} =$$

$$= \frac{a-2b+(-1)^k\sqrt{a^2-4ab+4b^2-4b+4ab-4b^2}}{2(1-a+b)} = \tag{19}$$

$$= \frac{a-2b+(-1)^k\sqrt{a^2-4b}}{2(1-a+b)},\ k=1,2.$$

From this we get

$$\vartheta_k + 1 = \frac{2-2a+2b+a-2b+\ (-1)^k\sqrt{a-4b}}{2(1-a+b)} = \frac{2-a+(-1)^k\sqrt{a^2-4b}}{2(1-a+b)}, \quad \text{k} = \ 1,2 \tag{20}$$

Thus the general solution to equation (16) has the form

$$Z_n = C_1(\vartheta_1+1)^n + C_2(\vartheta_2+1)^n, \tag{21}$$

where C_1 and C_2 are arbitrary constants. Then we will find the general solution of inhomogeneous equation (12) using the method of variation of constants, i.e. consider

$$Z_n = C_{1n}(\vartheta_1+1)^n + C_{2n}(\vartheta_2+1)^n. \tag{22}$$

Then

$$\begin{gathered} Z_n^{(')} = C_{1n}^{(')}((\vartheta_1+1)^n)^1 + C_{1n}^{(')}(\vartheta_1+1)^n + C_{1n}\vartheta_1(\vartheta_1+1)^n + \\ + C_{2n}^{(1)}\vartheta_2(\vartheta_2+1)^n + C_{2n}^{(1)}(\vartheta_2+1)^n + C_{2n}\vartheta_2(\vartheta_2+1)^n. \end{gathered} \tag{23}$$

Assuming that

$$C_{1n}^{(')}(\vartheta_1+1)^{n+1} + C_{2n}^{(')}(\vartheta_2+1)^{n+1} = 0, \tag{24}$$

and differentiating (23) once more and substituting into (12) we get

$$\begin{gathered} (1-a+b)\left[C_{1n}^{(')}\vartheta_1(\vartheta_1+1)^{n+1} + C_{1n}\vartheta_1^2\,(\vartheta_1+1)^n + \right. \\ \left. + C_{2n}^{(')}\vartheta_2(\vartheta_2+1)^{n+1} + C_{2n}\vartheta_2^2(\vartheta_2+1\ )^n\right] + \\ +(2b-\ a)\left[C_{1n}\vartheta_1(\vartheta_1+1)^n + C_{2n}\vartheta_2(\vartheta_2+1\ )^n\right] + \\ +b\left[C_{1n}(\vartheta_1+1)^n + C_{2n}(\ \vartheta_2+1\ )^n\right] = \ g_n. \end{gathered}$$

Thus, we get the following systems of algebraic equations:

$$\begin{cases} C_{1n}^{(')}(\vartheta_1+1)^{n+1} + C_{2n}^{(')}(\vartheta_2+1)^{n+1} = \ 0 \\ C_{1n}^{(')}\vartheta_1(\vartheta_1+1)^{n+1} + C_{2n}^{(')}\vartheta_2(\vartheta_2+1)^{n+1} = \frac{g_n}{1-a+b}, \end{cases} \tag{25}$$

$$\begin{gathered} W_n = \begin{vmatrix} (\vartheta_1+1)^{n+1} & (\vartheta_2+1)^{n+1} \\ \vartheta_1(\vartheta_1+)^{n+1} & \vartheta_2(\vartheta_2+1)^n+1 \end{vmatrix} = \\ = (\vartheta_2-\vartheta_1)(\vartheta_1+1)^{n+1}(\vartheta_2+1)^{n+1} \neq 0. \end{gathered} \tag{26}$$

Under conditions (26) according to Cramer's rule, from system (25) we get

$$C_{1n}^{(1)} = \frac{1}{W_n}\begin{vmatrix} 0 & (\vartheta_2+1)^{n+1} \\ \frac{g_n}{1-a+b} & \vartheta_2(\vartheta_2+1)^{n+1} \end{vmatrix} = \ -\frac{(\vartheta_2+1)^{n+1}}{W_n}\frac{g_n}{1-a+b} = \frac{-g_n(\vartheta_1+1)^{-n-1}}{(1-a+b)(\vartheta_2-\vartheta_1)},$$

$$C_{2n}^{(1)} = \frac{1}{W_n}\begin{vmatrix} (\vartheta_1+1)^{n+1} & 0 \\ \vartheta_1(\vartheta_1+1)^{n+1} & \frac{g_n}{1-a+b} \end{vmatrix} = \frac{(\vartheta_1+1)^{n+1}}{(\vartheta_2-\vartheta_1)(\vartheta_2+1)^{n+1}}\frac{g_n}{1-a+b} = \frac{g_n(\vartheta_2+1)^{-N-1}}{(1-a+b)(\vartheta_2-\vartheta_1)},$$

or taking into account (3), we have

$$C_{1n+1} - C_{1n} = \frac{-g_n(\vartheta_1+1)^{-n-1}}{(1-a+b)(\vartheta_2-\vartheta_1)},$$

$$C_{2n+1} - C_{2n} = \frac{g_n(\vartheta_2+1)^{-n-1}}{(1-a+b)(\vartheta_2-\vartheta_1)}.$$

From the last after summing over n, we have

$$C_{1n} = C_{10} - \sum_{k=0}^{n-1} \frac{g_k(\vartheta_1+1)^{-k-1}}{(1-a+b)(\vartheta_2-\vartheta_1)}, \tag{27}$$

$$C_{2n} = C_{20} + \sum_{k=0}^{n-1} \frac{g_k(\vartheta_2+1)^{-k-1}}{(1-a+b)(\vartheta_2-v_1)}.$$

Substituting (27) into (22) for inhomogeneous equation (12), we get the following general solution

$$Z_n = C_{10}(\vartheta_1+1)^n + C_{2n}(\vartheta_2+1)^n - \sum_{k=0}^{n-1} \frac{(\vartheta_1+1)^{n-1-k}}{(1-a+b)\,(\vartheta_2-\vartheta_1)} g_k +$$

$$+\sum_{k=0}^{n-1} \frac{(\vartheta_2+1)^{n-1-k}}{(1-a+b)\,(\vartheta_2-\vartheta_1)} g_k. \tag{28}$$

Thus, for fundamental solution of (12), we get the following expression

$$Z_{n,k} = Z_{n-k} = \begin{cases} \frac{(\vartheta_2+1)^{n-k-1}-(\vartheta_1+1)^{n-k-1}}{(1-a+b)(\vartheta_2-\vartheta_1)}, & k<n, \\ \quad 0\,, \quad k \geq n. \end{cases} \tag{29}$$

After substituting (29) into the left part of (12) we get

$$(1-a+b)Z^{('')}_{n-k} + (2b-a)Z^{(')}_{n-k} + bZ_{n-k} = \begin{cases} 0 \ k \neq n, \\ 1 \ \ k = n, \end{cases} \tag{30}$$

i.e.

$$l^* Z_{n-k} = \delta_{nk}, \tag{31}$$

where δ_{nk} is the Kronecker symbol.

Now, based on fundamental solution (29), we form expression (11).

$$\int_0^{N-1} \tilde{l} y_n \tilde{l} Z_{n-k} = (y_{N-k-1} Z_{N-1})^{(')} - (y_0 Z_{n-k})^{(')} +$$

$$+(a-2)\left[y_{N-1} Z^{(')}_{N-1-k} - y_0\ Z^{(')}_{k}\right] + a\left[y_{N-1} Z_{N-1-k} - y_0\ Z_{-k}\right] +$$

$$+\int_0^{N-1} y_n \left[(1-a+b)Z^{('')}_{n-k} + (2b-a)Z^{(')}_{n-k} + b\ Z_{n-k}\right] = \int_0^{N-1} f_n \tilde{l} Z_{n-k}.$$

Considering that $Z_{n-k} = 0$ when $n-k \leq 1$, we have

$$\begin{gathered} y_N Z_{N-k} - y_{N-1} Z_{N-1-k} + (a-2)\, y_{N-1}\left[Z_{N-k} - Z_{N-1-k}\right] + \\ + a y_{N-1} Z_{N-1-k} - \int_0^{N-1} f_n \left[Z^{('')}_{n-k} + 2Z^{(')}_{n-k} + Z_{n-k}\right] = -y_k, \quad k = \overline{0, N-2} \end{gathered} \tag{32}$$

Taking into account boundary condition (2) in (32), we obtain

$$y_k = \alpha y_0 Z_{N-k} - \beta y_1 Z_{N-k-1} + (a-2)\beta y_1 (Z_{N-k} - Z_{N-k-1}) +$$

$$+a\beta y_1 Z_{N-k-1} + \int_0^{N-1} f_n (Z_{n-k+2} - 2Z_{n-k+1} + Z_{n-k} + 2Z_{n-k+1} - 2Z_{n-k} + Z_{n-k})$$

or

$$y_k = [\alpha y_0 + (a-2)\beta\, y_1]\, Z_{N-k} + [a\, \beta y_1 - (a-2)\beta y_1 - \beta\, y_1]\, Z_{N-k-1} + \int_0^{N-1} f_n Z_{n-k+2}$$

or

$$y_k = [\alpha\, y_0 + (a-2)\,\beta\, y_1]\, Z_{N-k} + \beta y_1 Z_{N-k-1} + \sum_{n=0}^{N-2} f_n Z_{n-k+2}, \qquad k = \overline{0, N-2}.$$

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